

SPINOR METHODS

and, A Little More on Spin Dynamics

$$\frac{d\psi}{d\theta} = \frac{i}{2} (\vec{\Omega} \cdot \vec{\sigma}) \psi$$

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1 INTRODUCTION	3
2 PAULI MATRICES	4
3 SPIN TRANSPORT	6
HOME WORK 1: $e\vec{\omega} \cdot \vec{\sigma}\phi$	= $t_0 I + i\sigma_x t_x + i\sigma_s t_s + i\sigma_y t_y$
HOME WORK 2: Snake Mapping
4 SPIN MOTION NEAR AN ISOLATED RESONANCE	19

1 INTRODUCTION

- ◊ In dealing with Thomas-BMT equation of spin motion for spin- $\frac{1}{2}$ particles, in the first part of these lectures, spin was considered a classical quantity (by resorting to the principle of correspondence), handled under the form

$$\text{of a 3-vector in real space, } \vec{S} = \begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix}.$$

- ◊ An alternative method to deal with spin motion consists in using their spinor representation: a complex 2-vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

manipulated using spinor algebra: a 2×2 rotation matrix algebra.

- ◊ The complex components ψ_1 and ψ_2 of a spinor represent the respective probabilities of the $+\frac{1}{2}$ and $-\frac{1}{2}$ spin states (spin angular momentum $S = \pm \hbar/2$)
 - thus, in passing, normalization requires

$$|\psi|^2 \equiv \psi^\dagger \psi \equiv (\psi_1^*, \psi_2^*) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = |\psi_1|^2 + |\psi_2|^2 = 1$$

- ◊ We will only address 2-dimensional spinors, spin $\frac{1}{2}$ particles, however developments regarding 3-dimensional spinors, spin 1 particles, can be found in Conte-McKay (see bibliography).

2 PAULI MATRICES

- ◊ We sit in the usual moving frame (Serret-Frénet)

◊ Spinor matrix algebra has 4 basic elements:

the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_s = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

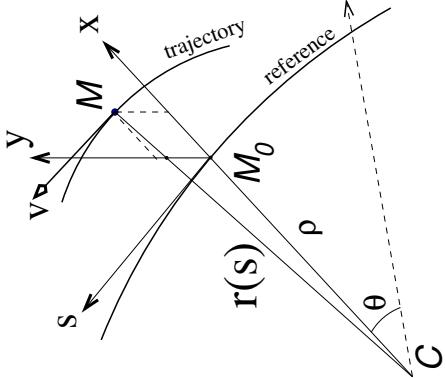
(indices relate to the frame axes, will see how)

$$\text{From spinor 2-vector } \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \text{ to classical 3-vector } \vec{S} = \begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix} :$$

define a 3-vector $\vec{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_s \\ \sigma_y \end{pmatrix}$, take $\psi^\dagger = (\psi_1^*, \psi_2^*)$ Hermitian conjugate of ψ ,

$$\text{then: } \vec{S} = \begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix} \equiv \psi^\dagger \vec{\sigma} \psi = \begin{pmatrix} \psi^\dagger \sigma_x \psi \\ \psi^\dagger \sigma_s \psi \\ \psi^\dagger \sigma_y \psi \end{pmatrix} = \begin{pmatrix} \psi_1 \psi_2^* + \psi_1^* \psi_2 \\ i\psi_1 \psi_2^* - i\psi_1^* \psi_2 \\ |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}$$

Note the expected $|\vec{S}|^2 = S_x^2 + S_s^2 + S_y^2 = 1$, as comes out with some algebra.



PAULI MATRIX PROPERTIES

- ◊ A variety of properties which are resorted to in various subsequent calculations:

$$\sigma_i^\dagger = \sigma_i \text{ (Hermitian)}; \quad \det(\sigma_i) = -1; \quad \operatorname{tr}(\sigma_i) = 0; \quad \sigma_i^\dagger \sigma_i = \sigma_i^2 = I \text{ (unitary)}$$

$$\sigma_x \sigma_s = -\sigma_s \sigma_x = i\sigma_y; \quad \sigma_s \sigma_y = -\sigma_y \sigma_s = i\sigma_x; \quad \sigma_y \sigma_x = -\sigma_x \sigma_y = i\sigma_s$$

$$\text{scalar product } \vec{\sigma} \cdot \vec{\sigma} = \vec{\sigma}^\dagger \cdot \vec{\sigma} = \vec{\sigma} \cdot \vec{\sigma}^\dagger = \sigma_x^2 + \sigma_s^2 + \sigma_y^2 = 3I$$

$$\text{vector product: } \vec{\sigma} \times \vec{\sigma} \equiv \begin{pmatrix} \sigma_x \\ \sigma_s \\ \sigma_y \end{pmatrix} \times \begin{pmatrix} \sigma_x \\ \sigma_s \\ \sigma_y \end{pmatrix} = 2i\vec{\sigma}$$

$$◊ \text{ A real-space 3-vector } \vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_s \\ \omega_y \end{pmatrix} \text{ can be represented by a } 2 \times 2$$

Hermitian matrix:

$$\vec{\sigma} \cdot \vec{\omega} = \vec{\omega} \cdot \vec{\sigma} = \omega_x \sigma_x + \omega_s \sigma_s + \omega_y \sigma_y = \begin{pmatrix} \omega_y & \omega_x - i\omega_s \\ \omega_x + i\omega_s & -\omega_y \end{pmatrix}$$

$$\det |\vec{\sigma} \cdot \vec{\omega}| = -\omega^2; \quad (\vec{\sigma} \cdot \vec{\omega})^n = \begin{cases} \omega^n I & \text{if n even} \\ \omega^{n-1} (\vec{\sigma} \cdot \vec{\omega}) & \text{if n odd} \end{cases} \quad (|\vec{\omega}| = \omega)$$

$$(\vec{\sigma} \cdot \vec{\omega}_a)(\vec{\sigma} \cdot \vec{\omega}_b) = I(\vec{\omega}_a \cdot \vec{\omega}_b) + i\vec{\sigma} \cdot (\vec{\omega}_a \times \vec{\omega}_b).$$

3 SPIN TRANSPORT

- An optical element may be represented by a 2×2 matrix (noted T , here). The transport of a spinor through that element writes

$$\psi_f = T(f \leftarrow i) \psi_i$$

with ψ_i and ψ_f the spinor respectively before and after the element.

- Assume the matrix T describes a spinor rotation by an angle $\phi \vec{\omega}$, with

$$\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_s \\ \omega_y \end{pmatrix} \text{ the precession vector (note } \omega = |\vec{\omega}| \text{)}.$$

T can thus be written under either one of four different forms:

$$\begin{aligned} T &= e^{\frac{i}{2}(\vec{\omega} \cdot \vec{\sigma})\phi} = I \cos \frac{\omega \phi}{2} + i \left(\frac{\vec{\omega}}{\omega} \cdot \vec{\sigma} \right) \sin \frac{\omega \phi}{2} \\ &= t_0 I + it_x \sigma_x + it_s \sigma_s + it_y \sigma_y = \begin{pmatrix} t_0 + it_y & t_s + it_x \\ -t_s + it_x & t_0 - it_y \end{pmatrix}, \end{aligned}$$

wherein, for consistency:

$$t_0 = \cos \frac{\omega \phi}{2}, \quad t_x = \frac{\omega_x}{\omega} \sin \frac{\omega \phi}{2}, \quad t_s = \frac{\omega_s}{\omega} \sin \frac{\omega \phi}{2}, \quad t_y = \frac{\omega_y}{\omega} \sin \frac{\omega \phi}{2}$$

◇ Note the properties:

$$\det(T) = t_0^2 + t_x^2 + t_s^2 + t_y^2 = 1; \quad \text{tr}(T) = 2t_0; \quad \underbrace{T^\dagger \neq T}_{\text{not Hermitian}}; \quad \underbrace{T^\dagger T = I}_{\text{unitary}}$$

- Examples

- (i) In a uniform vertical field $\vec{B} = B_y \vec{y}$, over an orbital section $\Delta\theta = \theta_2 - \theta_1$ (see home-work 2)
- spins precess around $\vec{B} \parallel \vec{y}$,
 - by an angle $\phi = G\gamma\Delta\theta$, (uniform field $\rightarrow \Delta\theta \equiv \alpha$)

thus

$$\phi \vec{\omega} = G\gamma\Delta\theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \phi \vec{\omega} \cdot \vec{\sigma} = G\gamma\Delta\theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_s \\ \sigma_y \end{pmatrix} = G\gamma\Delta\theta\sigma_y$$

yielding

$$\psi(\theta_2) = e^{\frac{i}{2}G\gamma\Delta\theta\sigma_y}\psi(\theta_1),$$

- (ii) Over one turn along the closed orbit in a perfect ring, in the moving frame, $\Delta\theta = 2\pi$, thus

$$\psi(\theta_2) = e^{\frac{i}{2}G\gamma 2\pi\sigma_y}\psi(\theta_1) = e^{\frac{i}{2}2\pi\nu_s\sigma_y}\psi(\theta_1),$$

$$T_{\text{1-turn}} = e^{\frac{i}{2}2\pi\nu_s\sigma_y}, \quad \nu_s = G\gamma \text{ spin tune.}$$

HOME WORK (optional): $e^{\frac{i}{e^2}(\vec{\omega} \cdot \vec{\sigma})\phi} = t_0 I + i\sigma_x t_x + i\sigma_s t_s + i\sigma_y t_y$

1/ Given $\vec{\omega} \cdot \vec{\sigma} = \omega_x \sigma_x + \omega_s \sigma_s + \omega_y \sigma_y$, $\omega = |\vec{\omega}|$,
 show that $(\vec{\omega} \cdot \vec{\sigma})^n = \begin{cases} \omega^n & \text{if n even} \\ \omega^{n-1}(\vec{\omega} \cdot \vec{\sigma}) & \text{if n odd} \end{cases}$

2/ With this property, using the Taylor expansion of $e^{\frac{i}{e^2}(\vec{\omega} \cdot \vec{\sigma})\phi}$,
 show that it can be written

$$e^{\frac{i}{e^2}(\vec{\omega} \cdot \vec{\sigma})\phi} = I \cos \frac{\omega \phi}{2} + \frac{i}{\omega} (\vec{\omega} \cdot \vec{\sigma}) \sin \frac{\omega \phi}{2}$$

3/ Using $\vec{\omega} \cdot \vec{\sigma} = \omega_x \sigma_x + \omega_s \sigma_s + \omega_y \sigma_y$, derive the 2×2 matrix form of
 $e^{\frac{i}{e^2}(\vec{\omega} \cdot \vec{\sigma})\phi}$

4/ Finally, identify with $T = t_0 I + i\sigma_x t_x + i\sigma_s t_s + i\sigma_y t_y$

- Transpose spinor 2-vector transport through an optical element,

$$\psi_f = T(f \leftarrow i) \psi_i$$

to 3D space spin 3-vector transport, using the coefficients t_0, t_x, t_s, t_y
of the 2×2 T-matrix:

$$\vec{S}_f = M(f \leftarrow i) \vec{S}_i = \begin{pmatrix} t_0^2 + t_x^2 - t_s^2 - t_y^2 & 2(t_x t_s + t_0 t_y) & 2(t_x t_y - t_0 t_s) \\ 2(t_x t_s - t_0 t_y) & t_0^2 - t_x^2 + t_s^2 - t_y^2 & 2(t_s t_y + t_0 t_x) \\ 2(t_x t_y + t_0 t_s) & 2(t_s t_y - t_0 t_x) & t_0^2 - t_x^2 - t_s^2 + t_y^2 \end{pmatrix} \vec{S}_i$$

with \vec{S}_i and \vec{S}_f the spin 3-vectors respectively at entrance and exit
of the optical element.

Rotations about reference frame axes

- Consider a rotation by an angle ϕ around the x axis,

$$\text{noting } \vec{\omega} = \vec{n}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

the unit x-rotation vector.

$$\text{Thus } \vec{\omega} \cdot \vec{\sigma} = \sigma_x.$$

- ◊ This x -axis spinor rotation is represented by the matrix (slide 6)

$$e^{\frac{i}{2}(\vec{\omega} \cdot \vec{\sigma})\phi} = T_x = e^{\frac{i}{2}\sigma_x\phi} = I \cos \frac{\phi}{2} + i \sigma_x \sin \frac{\phi}{2} = t_0 I + it_x \sigma_x = \begin{pmatrix} \cos \frac{\phi}{2} & i \sin \frac{\phi}{2} \\ i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}$$

- ◊ Transpose to 3D space using the T matrix coefficients (slide 9):

$$\text{rotation matrix } M_x = \begin{pmatrix} t_0^2 + t_x^2 & 0 & 0 \\ 0 & t_0^2 - t_x^2 & 2t_0 t_x \\ 0 & -2t_0 t_x & t_0^2 - t_x^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix}$$

which is the expected form for a ϕ angle rotation around the x axis.

- ◊ Repeat: get the spinor rotation matrix for the other 2 axes:

An s -axis spinor rotation is represented by the matrix

$$\boxed{T_s} = e^{\frac{i}{2}(\vec{n}_s \cdot \vec{\sigma})\phi} = e^{\frac{i}{2}\sigma_s\phi} = I \cos \frac{\phi}{2} + i\sigma_s \sin \frac{\phi}{2} = t_0 I + it_s \sigma_s = \begin{pmatrix} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} \\ -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix}$$

Question: calculate (or figure out) M_s

A y -axis spinor rotation is represented by the matrix

$$\boxed{T_y} = e^{\frac{i}{2}(\vec{n}_y \cdot \vec{\sigma})\phi} = e^{\frac{i}{2}\sigma_y\phi} = I \cos \frac{\phi}{2} + i\sigma_y \sin \frac{\phi}{2} = t_0 I + it_y \sigma_y = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}$$

Question: calculate (or figure out) M_y

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} :$$

- ϕ -rotation about an arbitrary axis, \vec{n}

$$T_n = e^{\frac{i}{2}(\vec{n} \cdot \vec{\sigma})\phi} = I \cos \frac{\phi}{2} + i(\vec{n} \cdot \vec{\sigma})\sin \frac{\phi}{2} = \begin{pmatrix} \cos \frac{\phi}{2} + in_y \sin \frac{\phi}{2} & (in_x + n_s) \sin \frac{\phi}{2} \\ (in_x - n_s) \sin \frac{\phi}{2} & \cos \frac{\phi}{2} - in_y \sin \frac{\phi}{2} \end{pmatrix}$$

- Transport through a sequence of optical elements

- take respective 2×2 spinor transport matrices T_1 (1st element) and T_2 (2nd element),

- take spinor states ψ_i and ψ_f before the first and after the second element:

$$\begin{aligned}\psi_f &= T_2 T_1 \psi_i = e^{\frac{i}{2}(\vec{\omega}_2 \cdot \vec{\sigma})} \phi_2 e^{\frac{i}{2}(\vec{\omega}_1 \cdot \vec{\sigma})} \phi_1 \psi_i. \\ &= \left[\cos \frac{\omega_2 \phi_2}{2} + i \left(\frac{\vec{\omega}_2}{\omega_2} \cdot \vec{\sigma} \right) \sin \frac{\omega_2 \phi_2}{2} \right] \left[\cos \frac{\omega_1 \phi_1}{2} + i \left(\frac{\vec{\omega}_1}{\omega_1} \cdot \vec{\sigma} \right) \sin \frac{\omega_1 \phi_1}{2} \right]\end{aligned}$$

The rules seen in slide 5 may be applied to develop this product.

- Or as well, use the t_0 , t_x , t_s , t_y coefficient notation (slide 6):

$$T_2 T_1 = (I t_{2,o} + i \sigma_x t_{2,x} + i \sigma_s t_{2,s} + i \sigma_y t_{2,y})(I t_{1,o} + i \sigma_x t_{1,x} + i \sigma_s t_{1,s} + i \sigma_y t_{1,y})$$

$$\begin{aligned}&= I(t_{2,o} t_{1,o} - t_{2,x} t_{1,x} - t_{2,s} t_{1,s} - t_{2,y} t_{1,y}) \\ &\quad + i \sigma_x (t_{2,o} t_{1,x} + t_{2,x} t_{1,o} - t_{2,s} t_{1,y} + t_{2,y} t_{1,s}) \\ &\quad + i \sigma_s (t_{2,o} t_{1,s} + t_{2,x} t_{1,y} + t_{2,s} t_{1,o} - t_{2,y} t_{1,x}) \\ &\quad + i \sigma_y (t_{2,o} t_{1,y} - t_{2,x} t_{1,s} + t_{2,s} t_{1,x} + t_{2,y} t_{1,o})\end{aligned}$$

- This generalizes to N optical elements:

$$\psi_f = T_N \dots T_2 T_1 \psi_i$$

- Example: insert a local field error in an otherwise perfect ring
 - ◊ We had (slide 7), for a y-axis spin precession by an angle $\phi = G\gamma\Delta\theta$ over an orbital section $\Delta\theta$ in uniform field:

$$\psi(\theta_2) = e^{\frac{i}{2}G\gamma\Delta\theta\sigma_y}\psi(\theta_1), \quad T(\Delta\theta) = e^{\frac{i}{2}G\gamma\Delta\theta\sigma_y}$$
 - ◊ Now, add a local field error
 - at orbital azimuth θ_e ,
 - causing spin to precess locally by angle ϕ_e around direction \vec{n}_e , so that
$$T_{\text{error}} = e^{\frac{i}{2}(\vec{n}_e \cdot \vec{\sigma})\phi_e}.$$
 - ◊ Thus, by virtue of the transport through a sequence of optical elements, the spinor transport matrix around the ring (“one-turn map”) writes

$$T_{1\text{-turn}} = T(\Delta\theta = 2\pi - \theta_e) T_{\text{error}} T(\Delta\theta = \theta_e)$$

$$= e^{\frac{i}{2}G\gamma(2\pi - \theta_e)\sigma_y} e^{\frac{i}{2}(\vec{n}_e \cdot \vec{\sigma})\phi_e} e^{\frac{i}{2}G\gamma\theta_e\sigma_y}$$

- Get precession angle and axis from the 2×2 matrix $T = e^{\frac{i}{2}(\vec{\omega} \cdot \vec{\sigma})\phi}$:

- We have seen earlier (slide 6) that, if one knows

$\frac{\vec{\omega}}{\omega}$
the precession angle $\omega\phi$ and the precession axis

then, the 2×2 spinor transport matrix can be written

$$T = e^{\frac{i}{2}(\vec{\omega} \cdot \vec{\sigma})\phi} = I \cos \frac{\omega\phi}{2} + i(\frac{\vec{\omega}}{\omega} \cdot \vec{\sigma}) \sin \frac{\omega\phi}{2}$$

- Now, conversely, if T is a spinor map, then

$$\boxed{\cos \frac{\omega\phi}{2} = \frac{1}{2} \operatorname{tr}(T),}$$

then, the precession angle $\omega\phi$ satisfies

$$\boxed{\frac{\vec{\omega}}{\omega} = \frac{-i}{2 \sin \frac{\omega\phi}{2}} \operatorname{tr}(T \vec{\sigma}) = \frac{-i}{2 \sin \frac{\omega\phi}{2}} \begin{pmatrix} \operatorname{tr}(T \sigma_x) \\ \operatorname{tr}(T \sigma_s) \\ \operatorname{tr}(T \sigma_y) \end{pmatrix}.}$$

HOME WORK (optional): SPINOR MAPS

1/ QUADRUPOLE LENS

Explicit the spinor map $(e^{\frac{i}{2}(\vec{\omega} \cdot \vec{\sigma})}\phi)$ for a quadrupole in the thin lens approximation, in terms of betatron amplitude, quadrupole strength, etc.

2/ PARTIAL SNAKE

Consider a localized spin rotator - let's call it a partial snake - at azimuthal angle θ_s along the reference orbit in a ring. This snake rotates the spin by an angle $a\pi$ ($a \leq 1$) around a longitudinal axis.

2.a/ Find its 2×2 spinor rotation matrix. Find its spin transport matrix in 3D space.

2.b/ Find the 1-turn spinor matrix. Find the fractional spin tune.

2.c/ Give a graph of $\text{frac}(\nu_{sp})$ as a function of $G\gamma$, over a 1-unit interval in $G\gamma$, for a few different values of a . Comment on the forbidden spin tune gap. What is the value of the spin tune gap width?

- Eigenvectors

- Note $T(\theta + 2\pi \leftarrow \theta) = T_{1\text{-turn}}$ the 1-turn spinor transport matrix;
- Let $\Lambda(\theta)$ be the 2-vector eigenvector. This writes

$\Lambda(\theta + 2\pi) = T_{1\text{-turn}} \Lambda(\theta)$ (this is the periodicity condition)
or equivalently

$$(T_{1\text{-turn}} - \lambda I)\Lambda = 0 \quad \text{eigenvalue equation}$$

- The two eigenvalues λ_{\pm} satisfy

$$\det(T_{1\text{-turn}} - \lambda I) = \lambda^2 - \lambda \operatorname{tr}(T) + \det(T) = 0$$

$$\text{wherein (slide 6)} \quad \operatorname{Tr}(T) = 2t_0 = \cos \frac{\omega\phi}{2}, \quad \det(T) = 1.$$

This yields

$$\lambda_{\pm} = t_0 \pm i\sqrt{1-t_0^2} = \cos \frac{\omega\phi}{2} \pm i \sin \frac{\omega\phi}{2} = e^{\pm i \frac{\omega\phi}{2}},$$

with $\omega\phi$ the spin precession angle over a turn.

Eigenvectors (continued)

- ◊ The eigenvectors result, i.e.:

$$\Lambda_{\pm} = \begin{pmatrix} it_{y,1-\text{turn}} \mp \sqrt{1 - t_{0,1-\text{turn}}^2} \\ -t_{s,1-\text{turn}} + it_{x,1-\text{turn}} \end{pmatrix}$$

- ◊ This can be transposed to real space 3-vector, using (slide 4)

$$\vec{n}_{\pm} = \Lambda_{\pm}^\dagger \vec{\sigma} \Lambda_{\pm}$$

which yields

$$n_{\pm} = \begin{pmatrix} \pm \frac{t_{x,1-\text{turn}}}{\sqrt{1 - t_{0,1-\text{turn}}^2}} \\ \pm \frac{t_{s,1-\text{turn}}}{\sqrt{1 - t_{0,1-\text{turn}}^2}} \\ \pm \frac{t_{y,1-\text{turn}}}{\sqrt{1 - t_{0,1-\text{turn}}^2}} \end{pmatrix}$$

- Differential equation of spin motion

- ◊ Spin motion satisfies the differential equation

$$\frac{d\psi}{d\theta} = \frac{i}{2}(\vec{\Omega} \cdot \vec{\sigma})\psi \quad \Leftrightarrow \quad \frac{d\vec{S}}{d\theta} = \vec{S} \times \vec{\Omega}$$

Following an oft-met notation, θ now denotes the trajectory deviation angle (velocity vector precession angle), not the orbital angle; $d\theta = 0$ in field-free sections.

- ◊ If $\vec{\Omega}$ does not depend on θ the spinor form is readily integrated:

$$\frac{d\psi}{\psi} = \frac{i}{2}(\vec{\Omega} \cdot \vec{\sigma})d\theta \quad \xrightarrow{\int_{\theta_1}^{\theta_2}} \quad \psi(\theta_2) = e^{\frac{i}{2}(\vec{\Omega} \cdot \vec{\sigma})(\theta_2 - \theta_1)} \psi(\theta_1)$$

This represents a spin rotation around $\vec{\Omega}$, by an angle $\vec{\phi} = \vec{\Omega}(\theta_2 - \theta_1)$.

- ◊ In a perfect ring, flat orbit, in the moving frame,

$$\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ G\gamma \end{pmatrix} \quad \text{so that} \quad \vec{\Omega} \cdot \vec{\sigma} = G\gamma \sigma_y;$$

- a vertical rotation axis - this is what σ_y tells

- spin precession angle in the interval $[\theta_1, \theta_2]$: $\phi = G\gamma(\theta_2 - \theta_1)$.

◊ If $\theta_2 - \theta_1 = 2\pi$ the particle completes a full revolution, $\phi / 2\pi = G\gamma$ is the number of spin precessions per turn, “spin tune”:

$$\nu_{sp} = G\gamma$$

4 SPIN MOTION NEAR AN ISOLATED RESONANCE

- Spin motion satisfies: $\frac{d\vec{S}}{d\theta} = \vec{S} \times \vec{\Omega}$;
- in the presence of perturbing fields the precession axis is no longer vertical:

$$\vec{\Omega} = \begin{pmatrix} \xi_R \\ -\xi_I \\ -G\gamma \end{pmatrix};$$

- the horizontal components Ω_x and Ω_z have been detailed in the previous lectures.

The opposite sign Ω_y component stems from clockwise reference frame rotation (here) versus anti-clockwise (here).

- In terms of spinors:

$$\frac{d\psi}{d\theta} = \frac{i}{2}(\vec{\Omega} \cdot \vec{\sigma})\psi = \frac{i}{2}(\xi_R \sigma_x - \xi_I \sigma_z - G\gamma \sigma_y)\psi$$

- Develop the resonance strength in Fourier series:

$$\xi = \xi_R + i\xi_I = \underbrace{(\epsilon_R + i\epsilon_I)}_{\text{Fourier series over period}} e^{-iG\gamma_n\theta}, \quad G\gamma_n = n \pm \nu_y$$

- ◇ Move to $G\gamma_n$ -frequency precessing frame (a change of variable which has merit of yielding a differential equation with constant coefficient):

$$\psi = e^{-\frac{i}{2}G\gamma_n\theta}\sigma_y \phi = \left(I \cos \frac{G\gamma_n\theta}{2} - i\sigma_y \sin \frac{G\gamma_n\theta}{2} \right) \phi,$$

- ◇ thus the new form

$$\frac{d\phi}{d\theta} = \frac{i}{2} \left[\epsilon_R \sigma_x - \epsilon_I \sigma_s - (G\gamma - G\gamma_n) \sigma_y \right] = \frac{i}{2} (\vec{\omega} \cdot \vec{\sigma}) \phi \quad \left[\vec{\omega} = \begin{pmatrix} \epsilon_R \\ -\epsilon_I \\ -\delta_n \end{pmatrix}, \delta_n = G\gamma - G\gamma_n \right],$$

readily integrable:

$$\frac{d\phi}{\phi} = \frac{i}{2} (\vec{\omega} \cdot \vec{\sigma}) d\theta \xrightarrow{\int_{\theta_1}^{\theta_2}} \phi(\theta_2) = e^{\frac{i}{2} (\vec{\omega} \cdot \vec{\sigma})(\theta_2 - \theta_1)} \phi(\theta_1)$$

$$\psi(\theta_2) = e^{-\frac{i}{2}G\gamma_n\theta_2\sigma_y} e^{\frac{i}{2}(\vec{\omega} \cdot \vec{\sigma})(\theta_2 - \theta_1)} e^{\frac{i}{2}G\gamma_n\theta_1\sigma_y} \psi(\theta_1)$$

Back to orbital frame:

- Introduce $T = t_0 I + it_x \sigma_x + it_s \sigma_s + it_y \sigma_y$ notations,

- note $\omega = \sqrt{|\epsilon_n|^2 + \delta_n^2}$

- and $\Delta\theta = \theta_2 - \theta_1$;

- ◊ after some algebra (use slides 6, 12) the transport matrix in the moving frame, near resonance, comes out:

$$\begin{aligned}
 T(\theta_2 \leftarrow \theta_1) = & I \left(\cos \frac{G\gamma_n \Delta\theta}{2} \cos \frac{\omega \Delta\theta}{2} - \frac{\delta_n}{\omega} \sin \frac{G\gamma_n \Delta\theta}{2} \sin \frac{\omega \Delta\theta}{2} \right) \\
 & + i\sigma_x \left(\frac{\epsilon_R}{\omega} \cos \frac{G\gamma_n(\theta_1 + \theta_2)}{2} \sin \frac{\omega \Delta\theta}{2} + \frac{\epsilon_I}{\omega} \sin \frac{G\gamma_n(\theta_1 + \theta_2)}{2} \sin \frac{\omega \Delta\theta}{2} \right) \\
 & + i\sigma_s \left(-\frac{\epsilon_I}{\omega} \cos \frac{G\gamma_n(\theta_1 + \theta_2)}{2} \sin \frac{\omega \Delta\theta}{2} + \frac{\epsilon_R}{\omega} \sin \frac{G\gamma_n(\theta_1 + \theta_2)}{2} \sin \frac{\omega \Delta\theta}{2} \right) \\
 & + i\sigma_y \left(-\frac{\delta_n}{\omega} \cos \frac{G\gamma_n \Delta\theta}{2} \sin \frac{\omega \Delta\theta}{2} - \frac{\sin G\gamma_n \Delta\theta}{2} \sin \frac{\omega \Delta\theta}{2} \right)
 \end{aligned}$$

- ◊ Note the double frequency:

- $G\gamma_n$, a high frequency at high $G\gamma$,
- and ω modulation $\rightarrow |\epsilon_n| \ll 1$ near resonance \sim slow frequency.

CASE OF AN INTEGER RESONANCE, $G_{\gamma_n} = n$

- ◊ In the previous expression for $T(\theta_2 \leftarrow \theta_1)$ take
 - $\theta_1 = 0$, $\theta_2 = m 2\pi$ with $m = \text{number of turns}$,
 - $G_{\gamma_n} = \text{integer}$ thus $\cos(G_{\gamma_n} m \pi) = \pm 1$ and $\sin(G_{\gamma_n} m \pi) = 0$.

◊ This results in:

$$T_{m\text{-turn}} = \pm \left[I \cos m\omega\pi + i\sigma_x \frac{\epsilon_R}{\omega} \sin m\omega\pi - i\sigma_s \frac{\epsilon_I}{\omega} \sin m\omega\pi - i\sigma_y \frac{\delta_n}{\omega} \sin m\omega\pi \right]$$

Note: with $G_{\gamma_n} = \text{integer}$, the G_{γ_n} frequency component vanishes, ω only is left.

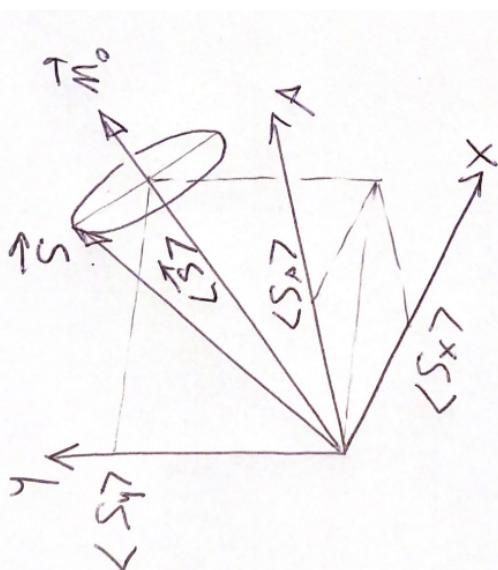
◊ Transpose to 3D space (using the t_0, t_x, t_s, t_y technique, slide 9):

$$\vec{S}(m 2\pi) = \begin{pmatrix} 2 \frac{\epsilon_R^2}{\omega^2} S + c & -2 \frac{\epsilon_R \epsilon_I}{\omega^2} S - \frac{\delta_n}{\omega} S & -2 \frac{\epsilon_R \delta_n}{\omega^2} S + \frac{\epsilon_I}{\omega} S \\ -2 \frac{\epsilon_R \epsilon_I}{\omega^2} S + \frac{\delta_n}{\omega} S & 2 \frac{\epsilon_I^2}{\omega^2} S + c & 2 \frac{\epsilon_I \delta_n}{\omega^2} S + \frac{\epsilon_R}{\omega} S \\ -2 \frac{\epsilon_R \delta_n}{\omega^2} S - \frac{\epsilon_I}{\omega} S & 2 \frac{\epsilon_I \delta_n}{\omega^2} S - \frac{\epsilon_R}{\omega} S & 2 \frac{\delta_n^2}{\omega^2} S + c \end{pmatrix} \vec{S}(0)$$

wherein, for short: $S = \sin^2(m\omega\pi)$, $s = \sin(2m\omega\pi)$, $c = \cos(2m\omega\pi)$.

- Each spin precesses, at frequency $\omega = \sqrt{|\epsilon_n|^2 + \delta_n^2}$, around an average 3-vector given by the average over turns:

$$\langle \vec{S}(m2\pi) \rangle = \left(\begin{array}{cc} \frac{\epsilon_R^2}{\omega^2} & -\frac{\epsilon_R\epsilon_I}{\omega^2} & -\frac{\epsilon_R\delta_n}{\omega^2} \\ -\frac{\epsilon_R\epsilon_I}{\omega^2} & \frac{\epsilon_I^2}{\omega^2} & \frac{\epsilon_I\delta_n}{\omega^2} \\ -\frac{\epsilon_R\delta_n}{\omega^2} & \frac{\epsilon_I\delta_n}{\omega^2} & \frac{\delta_n^2}{\omega^2} \end{array} \right) \vec{S}(0)$$



◊ The modulus $| \langle \vec{S}(m2\pi) \rangle |$ depends on the initial spin vector,

◊ $\langle \vec{S}(m2\pi) \rangle$ is the projection of $\vec{S}(m2\pi)$ on the periodic spin precession direction \vec{n}_\pm (slide 17).

- ◊ In particular if $\vec{S}(0) = (0, 0, S_y)$, the final polarization vector components are:

$$\langle S_x \rangle = n_x n_y = \frac{-\epsilon_R \delta_n}{\omega^2}, \quad \langle S_s \rangle = n_s n_y = \frac{\epsilon_I \delta_n}{\omega^2}, \quad \langle S_y \rangle = n_y = \frac{\delta_n^2}{\omega^2}.$$

- Express the periodic solution in terms of the resonance parameters:

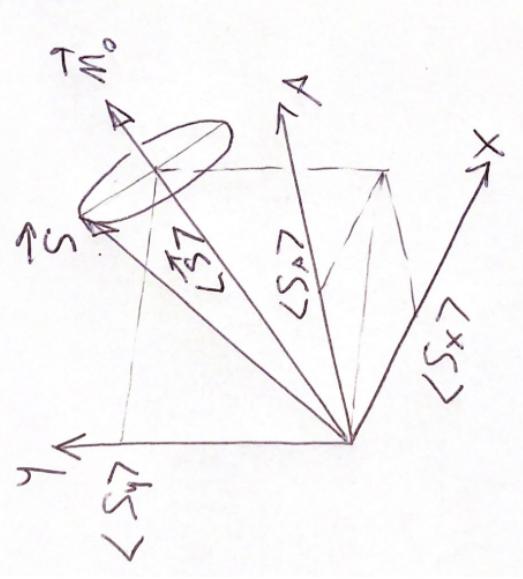
- ◊ Set $m=1$ in the m-turn matrix $T_{m\text{-turn}}$ (slide 22)

$$T_{1\text{-turn}} = \pm \left(I \cos \omega \pi + i \sigma_x \frac{\epsilon_R}{\omega} \sin \omega \pi - i \sigma_s \frac{\epsilon_I}{\omega} \sin \omega \pi - i \sigma_y \frac{\delta_n}{\omega} \sin \omega \pi \right)$$

that provides t_0, t_x, t_s, t_y .

Thus the two eigenvectors \vec{n}_0 of T are (slide 17)

$$\vec{n}_0 = \pm \begin{pmatrix} \frac{\epsilon_R}{\sqrt{|\epsilon_n|^2 + \delta_n^2}} \\ \frac{\sqrt{|\epsilon_n|^2 + \delta_n^2}}{\epsilon_I} \\ \frac{\delta_n}{\sqrt{|\epsilon_n|^2 + \delta_n^2}} \end{pmatrix}$$



- ◊ Far from the resonance:

$$|\delta_n| = |G\gamma - G\gamma_n| \rightarrow \infty, \text{ so } n_x, n_s \rightarrow 0, n_y \rightarrow 1, \vec{n}_0 \parallel \vec{y}.$$

- ◊ On the resonance:

$$|\delta_n| = 0, \text{ so } n_y = 0, \vec{n}_0 \text{ lies in the } (x, s) \text{ plane}$$

CASE OF AN INTRINSIC RESONANCE, $G\gamma_n = \mathbf{n} \pm \nu_y$

- ◊ In the earlier expression for $T_{m\text{-turn}}$ (slide 21), substitute

$$\theta_1 = 0, \quad \theta_2 = m 2\pi;$$

This yields:

$$\begin{aligned} T_{m\text{-turn}} &= I(\cos mG\gamma_n\pi \cos m\omega\pi - \frac{\delta_n}{\omega} \sin mG\gamma_n\pi \sin m\omega\pi) \\ &\quad + i\sigma_x \left(\frac{\epsilon_R}{\omega} \cos mG\gamma_n\pi \sin m\omega\pi + \frac{\epsilon_I}{\omega} \sin mG\gamma_n\pi \sin m\omega\pi \right) \\ &\quad + i\sigma_s \left(-\frac{\epsilon_I}{\omega} \cos mG\gamma_n\pi \sin m\omega\pi + \frac{\epsilon_R}{\omega} \sin mG\gamma_n\pi \sin m\omega\pi \right) \\ &\quad + i\sigma_y \left(-\frac{\delta_n}{\omega} \cos mG\gamma_n\pi \sin m\omega\pi - \sin mG\gamma_n\pi \sin m\omega\pi \right) \end{aligned}$$

- ◊ If desired (use to track for instance), transpose to 3D space spin transport matrix M (slides 9, 22)

$$\vec{S}(m 2\pi) = M \vec{S}(0).$$

- Precession of the spin vector \vec{S} :

Inspection of the T matrix (slide 25) shows that

- Spin vectors \vec{S} precess at frequency ω around \vec{n} , which precesses around the vertical axis with frequency $G\gamma_n$;
- The S_x and S_z components of \vec{S} oscillate
 - with an average zero value: $\langle S_x \rangle_{\text{turn}} = 0$ and $\langle S_z \rangle_{\text{turn}} = 0$, by contrast with the integer resonance case,
 - at frequency ω (precession frequency around \vec{n}),
 - modulated by a frequency $G\gamma_n$ (precession of \vec{n} around the vertical).
- The S_y component of \vec{S} oscillates at frequency ω around an $\vec{S}(0)$ -dependent average value

$$S_y = - \left(2 \frac{\epsilon_R \delta_n}{\omega^2} \mathcal{S} + \frac{\epsilon_I}{\omega} s \right) S_{0x} + \left(2 \frac{\epsilon_I \delta_n}{\omega^2} \mathcal{S} - \frac{\epsilon_R}{\omega} s \right) S_{0z} + \left(2 \frac{\delta_n^2}{\omega^2} \mathcal{S} + c \right) S_{0y}$$

thus a final polarization

$$\langle S_x \rangle = 0, \quad \langle S_z \rangle = 0, \quad \langle S_y \rangle = - \frac{\epsilon_R \delta_n}{\omega^2} S_{0x} + \frac{\epsilon_I \delta_n}{\omega^2} S_{0z} + \frac{\delta_n^2}{\omega^2} S_{0y}$$

- and note: the vertical component $\langle S_y \rangle$ has the same value as in the case of an integer resonance (slide 23).

- Spin precession vector:

- ◊ 1-turn spinor transport matrix:

Take an arbitrary turn, $\theta_1 = m2\pi$, and $\theta_2 = \theta_1 + 2\pi$, thus (slide 21):

$$\begin{aligned} T_{1\text{-turn}} &= I \left(\cos G\gamma_n \pi \cos \omega \pi - \frac{\delta_n}{\omega} \sin G\gamma_n \pi \sin \omega \pi \right) \\ &\quad + i\sigma_x \left(\frac{\epsilon_R}{\omega} \cos G\gamma_n \pi (2m+1) \sin \omega \pi + \frac{\epsilon_I}{\omega} \sin G\gamma_n \pi (2m+1) \sin \omega \pi \right) \\ &\quad + i\sigma_s \left(-\frac{\epsilon_I}{\omega} \cos G\gamma_n \pi (2m+1) \sin \omega \pi + \frac{\epsilon_R}{\omega} \sin G\gamma_n \pi (2m+1) \sin \omega \pi \right) \\ &\quad + i\sigma_y \left(-\frac{\delta_n}{\omega} \cos G\gamma_n \pi \sin \omega \pi - \sin G\gamma_n \pi \sin \omega \pi \right) \end{aligned}$$

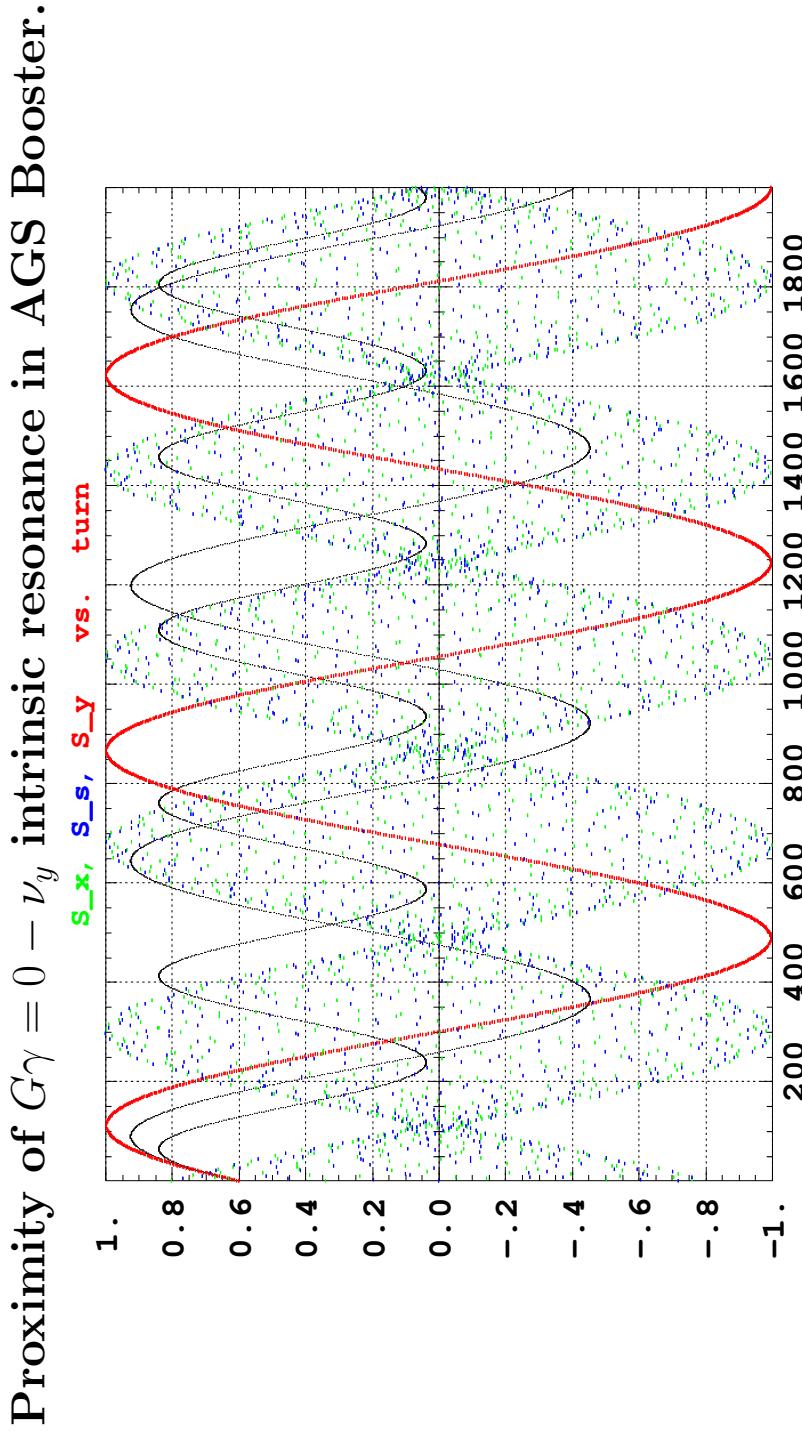
$$\vec{n}_\pm = \begin{pmatrix} \pm \frac{t_{x,1\text{-turn}}}{\sqrt{1-t_{0,1\text{-turn}}^2}} \\ \pm \frac{t_{s,1\text{-turn}}}{\sqrt{1-t_{0,1\text{-turn}}^2}} \\ \pm \frac{t_{y,1\text{-turn}}}{\sqrt{1-t_{0,1\text{-turn}}^2}} \end{pmatrix}$$

- ◊ The eigenvector writes (slide 17)

- its vertical component is $\propto t_y$, thus constant (independent of m):

\vec{n}_\pm precess around the y axis with frequency $G\gamma_n$.

- its x and s components oscillate with frequency $G\gamma_n$.



- ◊ Slow motion S_y spin component, on resonance (± 1 amplitude, red), is at frequency $\omega = 1/755[\text{turn}] = 0.001324$ oscillations per turn; $\delta_n = 0$ thus $|\epsilon_n| = 0.001324$.
- ◊ Rapidly oscillating are the S_x (green dots) and S_s (blue dots) components.
- ◊ The additional two slow motions are at distances to the resonance, respectively, $\delta_n = |\epsilon_n|$ and $\delta_n = 2|\epsilon_n|$. Frequencies satisfy $\omega = \sqrt{\delta_n^2 + |\epsilon_n|}$.

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